Model of propagation in random media

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Abstract

The problem examined is that of the propagation of a radar pulse in a random medium. The model developed takes into account, successively, the dispersivity of the medium, the influence of the earth's magnetic field and fluctuations in the electron density of the medium.

1. INTRODUCTION

The model described applies to propagation in ionized media. The first part looks into the effects of the Total Electron Content (TEC) along the path of propagation, the dispersivity of the medium and the influence of the magnetic field.

In the second part, the medium is described by the spectrum density of its index. The computations give the value of the two position two frequency autocorrelation function of the transmitted field. From this were deduced the distances of spatio-frequential coherence of the medium. The signal intensity scintillation rate was computed for a low fluctuation medium.

2. DISPERSIVITY OF THE MEDIUM

Since a plasma is a dispersive medium, a signal of finite duration may be considered as a sum of elementary waves which are propagated at different velocities. As they leave the medium under consideration, the various components recombine in a manner different to the original combination, resulting in a signal distortion which must be evaluated to establish the performance of radars whose beam on transmission and reception travels through a plasma.

The plasma can be compared to a filter whose frequency response is given by

$$H(\omega) = \exp(-\gamma(\omega))$$

with

$$\gamma(\omega) = \alpha(\omega) + j\,\beta(\omega)$$

The term $\alpha(\omega)$ characterizes losses by absorption through the plasma. The term $\beta(\omega)$ characterizes the variation in the phase velocity of the wave examined. If $E(\omega)$ is the spectrum of the transmitted signal, the spectrum of the signal after distortion from travelling through the dispersive medium is written :

$$S(\omega) = H(\omega)\, E(\omega)$$

The latter's temporal representation can be obtained by taking the reciprocal Fourier transform of $S(\omega)$:

$$s(t) = \frac{1}{2\pi} \int E(\omega)\; H(\omega) \exp(j\omega t)\, d\omega$$

Before detection, the received signal goes into a matched filter which is designed to maximize the output signal to noise ratio and to carry out the compression function when the signal sent is coded.

After passing through a plasma, the received signal is no longer the exact copy of the signal sent and the matched filter ceases to function properly. This malfunction may become particularly apparent with wide-band coded signals. From the hypothesis of white noise in the reception band, the theory shows that the temporal response of the matched filter is given by the formula :

$$p(t) = \int s(\tau) * e(\tau - t)\, d\tau$$

From the frequential point of view, the output signal spectrum from the filter is written :

$$S'(\omega) = K(\omega)\ K_0^*(\omega)$$

$K(\omega)$ and $K_0(\omega)$ are the spectrums of transmitted and received signals $S(t)$ and $e(t)$. The temporal response of the matched filter is thus given by the formula :

$$p(t) = \frac{1}{2\pi} \int K(\omega)\ K_0^*(\omega) \exp(j\omega t)\, d\omega$$

In the event that s (t) is the signal that has travelled through the plasma, one may write :

$$p(t) = \frac{1}{2\pi} \int |E(\omega)|^2\ H(\omega) \exp(j\omega t)\, d\omega$$

Depending on the analytical form of the signals studied, either the time or frequency representations may prove more advantageous in calculating the output signal from the matched filter, as both representations are equivalent from a theoretical point of view.

If we restrict ourselves to the case of a collision-free plasma and in the absence of magnetic field, attenuation by absorption may be considered as negligible and the only relevant factor is dispersion caused by phase velocity variation in the electromagnetic wave depending on frequency. One may therefore write :

$$H(\omega) = \exp(-j\,\beta(\omega))$$

In a collision-free plasma, the index value is :

$$n = \left(1 - (f_c / f)^2\right)^{1/2}$$

f_c is the frequency (critical frequency) of the plasma, whose corresponding pulsation is :

$$\omega_c = N e^2 / m \varepsilon_0$$

N is the number of electrons per volume unit.

If we assume that carrier frequency is much higher than resonance frequency and that the signal pass-band is weak in comparison to the latter, it is legitimate to approximate the term $\beta(\omega)$ characteristic of the dispersion of the medium by a second order Taylor series expansion :

$$\beta(\omega) = \beta(\omega_0) + \beta'(\omega_0)(\omega - \omega_0) + \frac{\beta''(\omega_0)(\omega - \omega_0)^2}{2}$$

where ω_0 is the carrier frequency pulsation.

The terms β' and β'' can be obtained by considering the optical path which is given in first approximation by :

$$\Delta S = \int_{s_1}^{s_2} \left(1 - \frac{1}{2}\left(\frac{f_c}{f}\right)^2\right) ds$$

In these conditions, the phase rotation on the path can be written in the light of the formula which gives the plasma resonance frequency :

$$\Delta\phi = \frac{\omega}{c}(s_2 - s_1) - \frac{e^2 N_T}{2 m \varepsilon_0 c \omega}$$

In this expression, the first term corresponds to the phase shift of a wave normally propagated in a vacuum, while the second term gives the additional phase shift associated with the presence of plasma. N_T is the total electron content. By derivation with respect to ω we obtain the term β' of the development of β(ω) :

$$\beta' = \frac{d\Delta\phi}{d\omega} = \frac{s_2 - s_1}{c} + \frac{e^2 N_T}{2 m \varepsilon_0 c \omega_0^2} \; .$$

This magnitude has the dimension of a time : the first term corresponds to the normal delay associated with the propagation of electromagnetic waves in a vacuum, and the second term to the delay associated with group velocity in plasma.

By a further derivation with respect to ω we obtain :

$$\beta'' = \frac{d^2\Delta\phi}{d\omega^2} = -\frac{e^2 N_T}{2 m \varepsilon_0 c \omega_0^3}$$

This magnitude has the dimensions of a time squared.

In the context of these approximations, the temporal representation of the signal after passing through the plasma is given by :

$$s(t) = \exp(j\omega_0 t)\, s_1(t^*)$$

and the output of the matched filter is given by:

$$p(t) = \exp(j\omega_0 t)\, p_1(t^*)$$

with $t^* = t - \beta'$

$$s_1(t) = \frac{1}{2\pi}\int E(\omega + \omega_0) \exp(j(\beta''\omega^2/2 + \omega t)\, d\omega$$

$$p_1(t) = \frac{1}{2\pi}\int \left|E(\omega + \omega_0)\right|^2 \exp(j(\beta''\omega^2/2 + \omega t)\, d\omega$$

The effect of the plasma is thus characterized by the terms $\beta'(\omega_0)$ and $\beta''(\omega_0)$. The first reflects the group delay linked with propagation in the medium and the second reflects dispersivity. These two terms are expressed solely and in a simple manner in relation to the total electron content, which can be easily computed once the plasma electron distribution is known.

3. INFLUENCE OF THE MAGNETIC FIELD

The properties of the plasma are profoundly altered in the presence of a magnetic field. Apart from the resonance frequency f, which was introduced previously during the study of radar signal distortion by a plasma in the absence of magnetic field, we need to consider the cyclotron frequency given by :

$$f_b = \frac{e H_0}{2 \pi m}$$

f_b = 1.4 Mhz for the earth's magnetic field.

This frequency corresponds to electron rotation which, in the presence of a magnetic field, describe a circular trajectory under the influence of the electrical field. The medium becomes doubly refractive and two circular polarized waves are propagated with different velocities according to their direction of polarization. A wave whose electric field vector turns in the reverse direction to the direction of rotation of the electrons in the plasma is called an ordinary wave. Otherwise, the wave is called extraordinary.

We know that any wave in plane polarization can be broken down into two circular waves polarized in the reverse direction to each other. The result of this is that as the two waves travel through a plasma in the presence of a magnetic field, they will have two different phase angle rotations. As they leave the plasma, these two waves recombine to give a wave with plane polarization but whose electrical field vector has changed by a certain angle compared with the direction of the original vector : this phenomenon is characteristic of the Faraday effect. At high frequencies the angle of Faraday rotation is given by the formula :

$$\vartheta(\omega) = \frac{\omega_b}{2 c \omega^2} \int_{s_1}^{s_2} \omega_c^2 \cos \gamma \, ds$$

γ is the angle between the magnetic field and the direction of propagation. In first approximation, it is legitimate to remove γ from the integral on the basis that this angle remains constant on the path of a ray.

On the basis of this approximation, the Faraday angle can be expressed simply in accordance with the total electron content :

$$\vartheta(\omega) = \frac{e^2 \omega_b}{2 m c \varepsilon_0 \omega^2} N_T \cos \gamma$$

Since the frequencies under consideration are always large in relation to the plasma's cut-off frequency, an approximation of $\vartheta(\omega)$ can legitimately be performed by a first order Taylor series expansion :

$$\vartheta(\omega) = \vartheta(\omega_0) + \vartheta'(\omega_0)(\omega - \omega_0)$$

By deriving the expression which gives the Faraday rotation, we obtain :

$$\vartheta'(\omega) = \frac{e^2 \omega_b}{2 m c \varepsilon_0 \omega^3} N_T \cos \gamma$$

The Faraday effect is particularly apparent for low frequencies of the band in question, which in practical terms necessitates circular polarized reception. As the plasma's electron density increases, the Faraday effect plays an increasingly significant role, even at high frequencies.

Thus, at an emission frequency of 4GHz, for a total electron content equal to 100 times that of a normal ionosphere, the Faraday rotation is 2.2 radians (126 degrees), which involves circular polarized reception.

In the case of circular polarized reception, the distortion of the radar signal in the presence of the earth's magnetic field is the same as in the absence of a magnetic field. It thus depends only on the group delay and on the plasma's dispersivity term. The previously established formalism remains valid.

In the case of plane polarization reception, the temporal response on leaving the matched filter is written :

$$p(t) = \frac{1}{2\pi} \int \cos(\vartheta(\omega)) \left|E(\omega)\right|^2 H(\omega) \exp(j\omega t)\, d\omega$$

E (ω), ϑ (ω), H (ω) are the spectrum of the transmitted signal, the Faraday rotation angle and the plasma's frequency response.

By using the same computation technique as in the absence of a magnetic field, for the matched filter's temporal response we obtain the expression :

$$p(t) = \frac{e^{j(\omega_0 t - \beta_0)}}{4\pi} \left[e^{j\vartheta_0} J(t^* + \vartheta') + e^{-j\vartheta_0} J(t^* - \vartheta') \right]$$

with :

$$J(t) = \int \left|E(\omega + \omega_0)\right|^2 \exp\left(-j\left(\frac{\beta'' \omega^2}{2} - \omega t\right)\right) d\omega$$

This formula can be interpreted from a physical point of view by observing that it corresponds to two pulses 2 ϑ' apart : one has a group delay equal to $\beta' + \vartheta'$ and the other a group delay equal to $\beta' - \vartheta'$. These two pulses result from the plane polarized wave splitting into two circular polarized waves which are propagated at different velocities, one corresponding to the ordinary wave and the other to the extraordinary wave.

By way of example, the formalism established above is applied to an unmodulated Gaussian pulse. The pulse considered has the form :

$$e(t) = \exp(-bt^2)\exp(j\omega_0 t)$$

This is an HF impulse which has a frequency of f_0 and a Gaussian amplitude.

After passing through the plasma in the presence of the earth's magnetic field, the signal emerging from the matched filter is given by :

$$p(t) = R(t^*) \exp(j\Psi(t^*))$$

with

$$R(t) = \frac{1}{2}\sqrt{\frac{\pi}{b}} \frac{e^{-\frac{b\vartheta^{1/2}}{2\alpha}}}{\alpha^{1/4}} e^{-\frac{bt^2}{2\alpha}} \left(\mathrm{ch}(2U_1) + \cos(2\Phi_1)\right)^{1/2}$$

$$\Psi(t) = \Psi_0 + \left(\Psi_0 + \frac{b^2\beta''\vartheta'}{\alpha}\right)t + \frac{b^2\beta''}{2\alpha}t^2 - \mathrm{arctg}\frac{\sin 2\Phi_1}{\exp(-2U_1) + \cos(2\Phi_1)}$$

$$\alpha = 1 + (b\beta')^2 \quad ; \quad U_1 = b\beta' t / \alpha \quad ; \quad \Phi_1 = \frac{b^2\beta''\theta'}{\alpha}t - \theta_0$$

$$\Psi_0 = \omega_0\beta' - \beta_0 - \frac{1}{2}\mathrm{arctg}\, b\beta'' + \frac{b^2\beta''}{2\alpha}\vartheta'^2$$

These formulae show that travelling through plasma in the presence of a magnetic field has the following effects :

~ the signal envelope undergoes an amplitude distortion in relation to the Gaussian form ;
~ the maximum is attenuated ;
~ the phase follows a time-dependent complex modulation law ;
~ there exists an additional phase rotation linked to the group delay, to the dispersal term and the Faraday effect.

In addition to these effects, a further global factor is the group delay time as with the absence of the earth's magnetic field.

To illustrate this point, figure 1 shows a Gaussian impulse with a carrier frequency of 150 Mhz for an electron density equal to 10 times the electron density of the normal ionosphere. The pass-band is 2 Mhz.

The Faraday rotation angle then has an average value of 157.5 radians, which corresponds to a number of turns of the electrical field vector such that it is legitimate to consider the angle θ_0 as random and equally distributed between 0 and 2π. This is taken as the parameter. For $\theta_0 = 0°$ the impulse is symmetrical but may be considerably distorted (diminution of the maximum amplitude and stretching over time). For $\theta_0 = 45°$ the impulse is asymmetric. For $\theta_0 = 90°$ the impulse is split with a passage through zero.

4. INCLUDING FLUCTUATIONS

A statistical model can be developed by including fluctuations in the medium's electron density. This model has the advantage of providing analytical results for a certain number of problems linked with propagation in ionized media. The parameters for these computations are set in relation to the standard phase deviation of the signal transmitted, which represents the extent of turbulence in the medium's electron density. The phase variance can be computed with the expression :

$$\sigma_\Phi^2 = 2(\lambda r_e)^2 L L_0 \sigma_{Ne}^2$$

where L is the propagation distance within the ionized medium,
L_0 is the outer scale of the inhomogeneities inside the medium,
σ_{Ne} i s the standard deviation of the electron density,
$r_e = 2.82.\ 10^{-15}$ is the electron radius.

The problem data which enable the computations to be made are L, L_0, σ_{Ne} and the medium index spectrum density. Because of the influence of the earth's magnetic field, the ionized medium is anisotropic. The spectrum density is expressed in accordance with a vectorial wave number. This wave number is scalar if the medium is isotropic. Two models of spectrum density can be used, the latter being approximated either by a power function or by Bessel functions $K_0(k)$. These two models, largely equivalent for low values of the k argument, are used to develop analytical computations. Depending on the problem studied, one or other of these formulations is the best suited.

5. COHERENCE FUNCTION OF THE TRANSMITTED FIELD

As regards signal propagation, the medium is characterized by the autocorrelation function Γ of the field transmitted, usually designated as the coherence function.

$$\Gamma(\rho_1, \rho_2, k_1, k_2, z) = \langle U(\rho_1, k_1, z)\, U^*(\rho_2, k_2, z)\rangle$$

This function verifies the propagation equation which for a random medium can be written in the following form :

$$\frac{\partial \Gamma}{\partial z} = -\frac{j}{2}\left(\frac{1}{k_1^2}\nabla_{t1} - \frac{1}{k_2^2}\nabla_{t2}\right)\Gamma - 2\pi^2 r_e^2\left[\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)A_{\Delta N}(0) - \frac{2}{k_1 k_2}\,A_{\Delta N}(\rho)\right]$$

In this expression, A is the integral of the autocorrelation function of the medium index along the line of propagation.

The corresponding solution can be obtained both in the case of plane wave and spherical wave approximation.

Plane wave type solution

In the case of plane wave type propagation, there is zero difference in the transverse Laplacians. The equation is limited to the first and third terms. As a result, we obtain :

$$\Gamma = \exp\left[-2\pi^2 r_e^2 z\left[\left(\frac{1}{k_1} - \frac{1}{k_2}\right)^2 A_{\Delta N}(0) + \frac{2}{k_1 k_2}\left(A_{\Delta N}(0) - A_{\Delta N}(\rho)\right)\right]\right]$$

i.e. depending on the phase variance and the structure function :

$$\Gamma_3 = \exp\left[-\left[\frac{\sigma_\Phi^2\,\omega_d^2}{2\,\omega_0^2} + \frac{D_\Phi(\rho, z)}{2}\right]\right]$$

The inclusion of the anisotropy of the medium is reflected in a multiplicative geometric factor in the structure function.

General solution

The general solution of the propagation equation can be obtained without using the plane wave approximation hypothesis. In this case, the space variables are changed by means of the sum and difference of the coordinates. By separating the variables, it can be shown that the solution is expressed as the product of three functions. The first corresponds to the solution obtained by the plane wave approximation. The other functions are solutions of differential equations deduced from the propagation equation.

The numerical models constructed show that for frequencies higher than the GHz, the general solution is not significantly different from the solution corresponding to the plane wave approximation.

6. IMPULSE RESPONSE OF THE MEDIUM

The impulse response of the medium is the Fourier transform of the coherence function.

$$R(t) = \frac{1}{2\pi} \int \Gamma(\omega_d, \rho, z)\ \exp(-j\,\omega_d t)\ d\omega_d$$

In the case of plane wave approximation, we obtain :

$$R(t) = \frac{\omega_{coh}}{2\sqrt{\pi}}\ \Gamma(\rho, z)\ \exp(-(\omega_0 t / \sigma_\Phi \sqrt{2})^2)$$

with $$\omega_{coh} = \omega_0 \sqrt{2} / \sigma_\Phi$$

The curves shown on figure 2 are plotted in reduced coordinates Γ / ω_{coh} and $t * \omega_{coh}$. The peak value is $1 / (2 \sqrt{\pi})$, i.e. 0.28. They are obtained with the expression of Γ corresponding to its general shape (spherical wave) for five values of the standard deviation of the phase of the signal transmitted, from 5 radians to 7500 radians. The increase by σ_ϕ is reflected in a flattening of the curve : reduction in the peak value and increase in the width. The width at mid height varies from 2 for 5 radians to 9 for 7500 radians. For a medium with a zero standard phase deviation, the impulse response obtained is a Dirac delta function corresponding to a medium with an infinite pass-band.

7. COHERENCE OF THE MEDIUM

The coherence distance and the coherence band are deduced from the coherence function. They are defined for the value $\Gamma = \exp(-1)$, i.e. 4.3 db. For $\omega_d = 0$, the coherence distance ℓ_{coh} is such that $D_\Phi(\ell_{coh}, z) = 2$.

Likewise, for $\rho = 0$, the coherence band is equal to $\omega = \omega_{coh}$.

The duration of temporal coherence in first approximation is equal to the quotient of the spatial coherence distance ℓ_{coh} divided by the drift velocity of the medium.

By designating $H(\omega)$ as the transmitted signal spectrum, the time signal received by the reception antenna is :

$$Y(t, \rho) = 2\pi \int \Gamma(\omega, z) \left| H(\omega) \right|^2 \exp(j\,\omega t)\, d\omega$$

If the width of the incident spectrum is smaller than the value of the medium's coherence band, the result is a deterioration in the signal received which in this case leads to a temporal widening and a distortion in the output signal.

8. SCINTILLATION RATE

The square of the scintillation rate is by definition equal to the intensity scintillation variance, centred and scaled. It is calculated from the expression of the fourth order moment Γ_4 of the transmitted field. The latter satisfies the equation below deduced from the propagation equation:

$$\frac{\partial \Gamma_4}{\partial z} = -\frac{j}{k} \nabla_{r1} . \nabla_{r2}\, \Gamma_4 - \frac{1}{2L} F(r_1, r_2)\, \Gamma_4$$

L is the distance travelled inside the medium,

F is expressed by means of the structure function of the medium's index,

r_1 and r_2 are two vectors in the plane transverse to the direction of propagation.

This equation can be solved numerically either by a finite difference technique, or by a split step technique. In the latter case, the solution for each step is obtained after a Fourier transformation of Γ_4 in relation to the space dimensions. In the case of weak fluctuations, an approximated value of S_4^2 is provided by the integral :

$$S_4^2 = 4\int \gamma_\Phi(K) \sin^2\left(K^2 z^* / 2 k \cos \vartheta\right) dK$$

with z^* such that $1 / z^* = 1 / z_t + 1 / z_r$

z_r and z_t are the distances from the ionized medium to the reception point and to the observation point.

The angle ϑ is the angle at the zenith. $\gamma_\phi(K)$ is the spectrum density of the phase of the signal transmitted in the medium. After a change of variables and according to the phase variance of the transmitted signal, we obtain :

$$S_4^2 = \frac{8\sqrt{2}\,\sigma_\Phi^2\,\Gamma(p/2)}{\sqrt{\pi}\,\Gamma\left((p-1)/2\right)\xi^{1/2}} \int \frac{\sin^2 V^2}{\left(1 + 2V^2/\xi\right)^{P/2}} dV.$$

with $\xi = z^* / \left(k L_0^2 \cos\theta\right)$

This integral is calculated numerically. In the case of an anisotropic medium, the integral to be computed is :

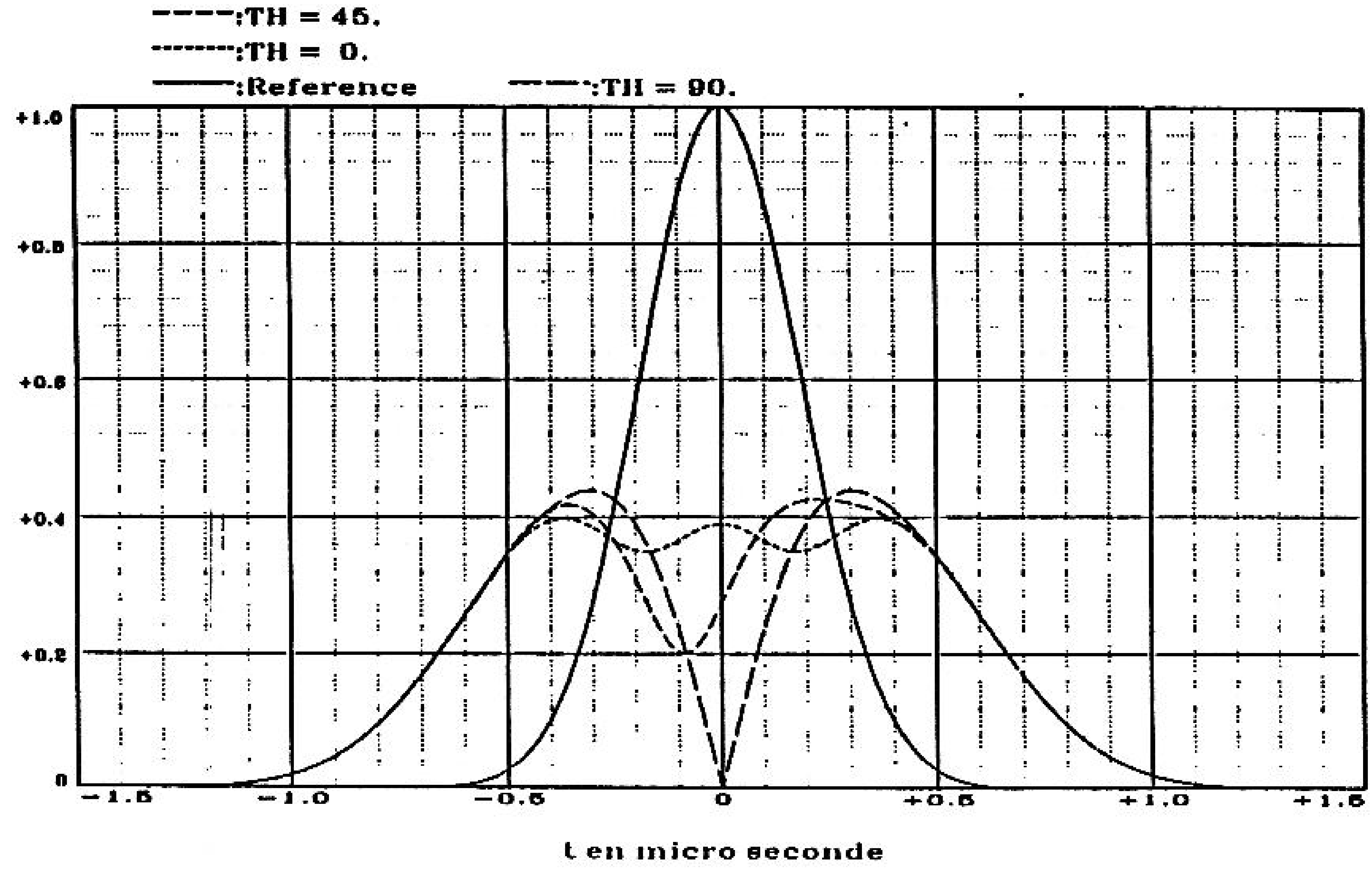
$$S_4^2 = \frac{4abC_P}{(2\pi)^2} \iint \frac{\sin^2(K^2 Z)}{\left[q_0^2 + A K_x^2 + B K_x K_y + C K_y^2\right]^{p/2}} dK$$

where a and b are the average dimensions of the large and small axis of the ellipses forming the irregularities of the medium. This expression is calculated by ignoring $q_0^2 = 4\pi^2 / L_0^2$ in front of the other denominator terms, given that the essential contribution to the scintillation rate comes from low values of K. The calculation is performed after the vector K has been rotated in order to remove the term in $K_x K_y$, and then shifting to polar coordinates. The integrals obtained are also calculated numerically.

The simulation results are presented in figure 3 for values of a and b set respectively at 5 km and 3 km. In the light of the computation assumptions, the values of S_4 obtained in the case of an anisotropic medium are only meaningful for high values of L_0.

9. CONCLUSION

The chief mechanisms affecting the propagation of impulse signals in an ionized medium were studied. Priority was given to analytical calculations in the formalism developed in order to put a figure on the significance of the phenomena depending on the parameters of the problem.

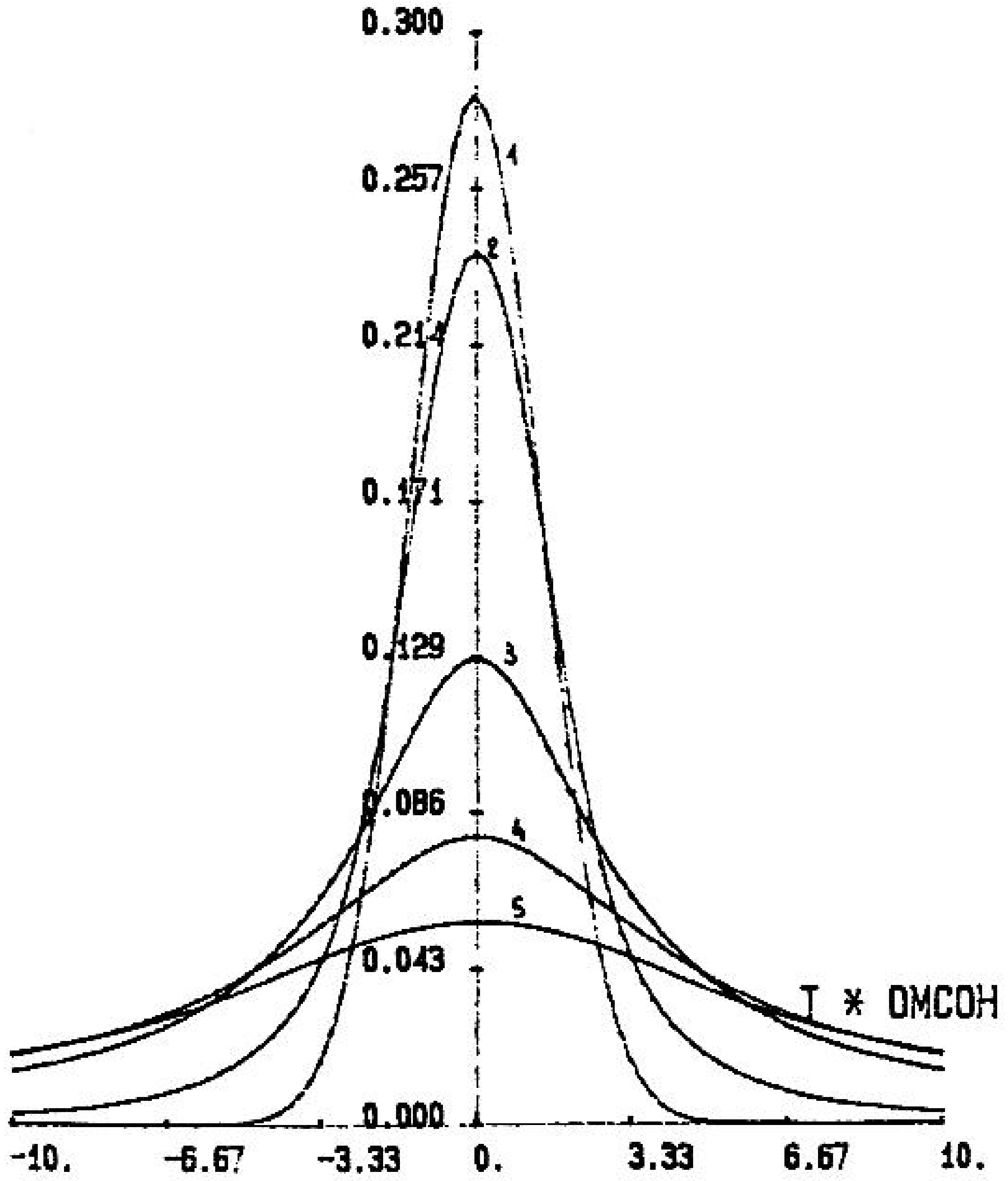


Réponse du filtre adapté

Présence du champ magnétique terrestre

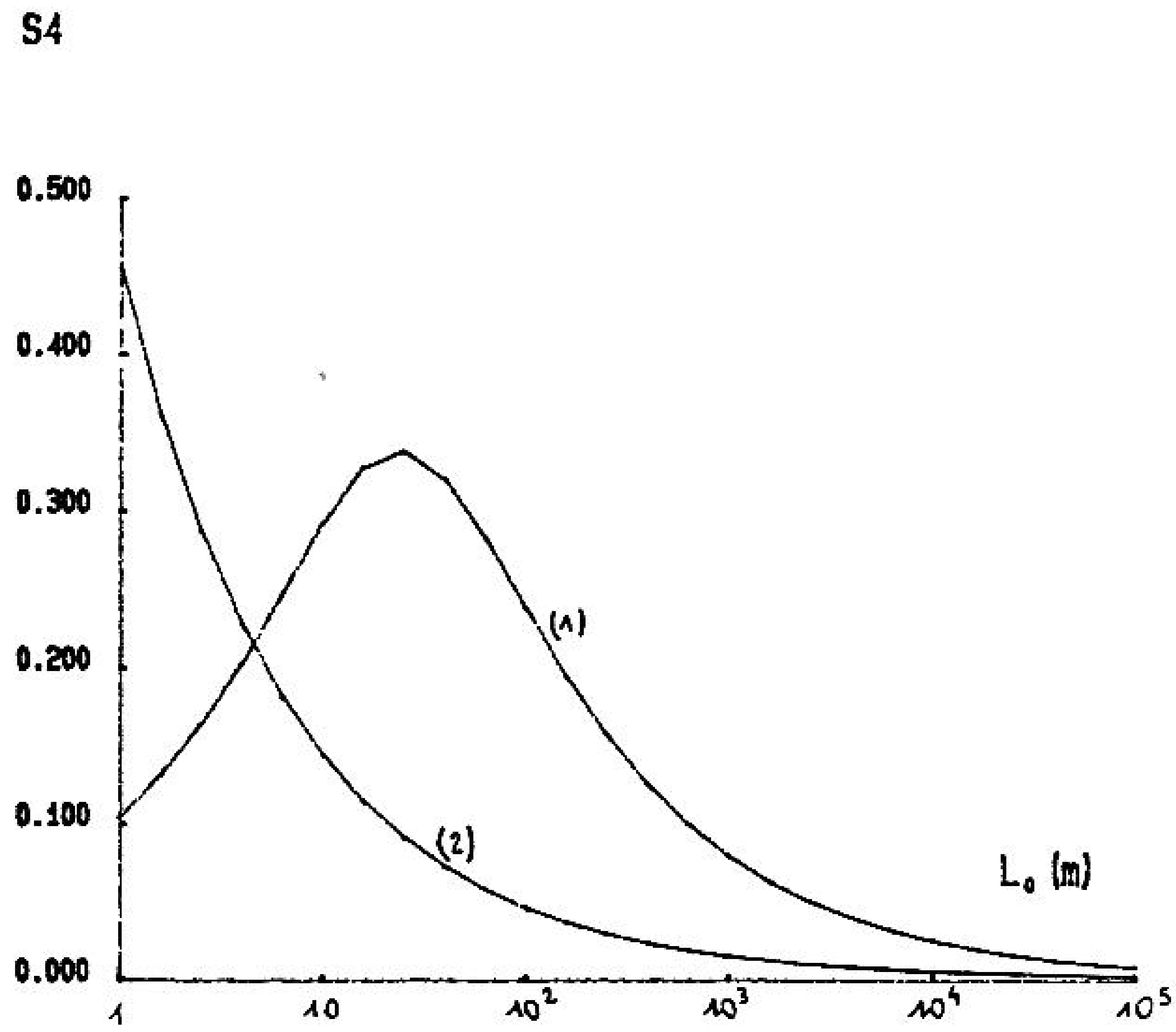
f_0 = 150 MHz ; Δf = 2 MHz ; Densité électronique = 10 * ionosphère

figure 1



réponse impulsionnelle du milieu
pour des valeurs σ_{Φ} de l'écart type de la phase égales à
5 r. (1), 500 r. (2), 2500 r. (3), 5000 r. (4), 7500 r. (5)

figure 2



Taux de scintillation du signal transmis
en fonction de la taille des inhomogénéités du milieu

milieu isotrope (1) - milieu anisotrope (2)

figure 3